Spectral dimension of a wire network

1985 J. Phys. A: Math. Gen. 18 2807

(http://iopscience.iop.org/0305-4470/18/14/030)

J. Phys. A: Math. Gen. 18 (1985) 2807-2823. Printed in Great Britain

Spectral dimension of a wire network

Hiroshi Watanabe
Department of Mathematics, Tokyo Metropolitan University, Tokyo 158, Japan

Received 2 January 1985, in final form 26 March 1985

Abstract. By a rigorous analysis for diffusion on a wire network, the spectral dimension of a fractal is shown to be independent of the local structure. It is discussed that diffusion does not occur when the lattice spacing tends to zero on such a fractal that the spectral dimension is less than the Hausdorff dimension if a free field on the fractal exhibits a certain long distance behaviour. For a Sierpinski carpet, the spectral dimension is evaluated within the bond-moving approximation (Migdal-Kadanov renormalisation). As a result, we obtain a value smaller than the Hausdorff dimension.

1. Introduction

Fractal structures are characterised by dilation symmetry in contrast to Euclidean spaces with translational invariance. Recently, there has been a good deal of interest in a fractal space (Rammal 1984) mainly (i) as a situation to reconsider our familiar notions, e.g. dimensionality (Alexander and Orbach 1982, Rammal and Toulouse 1983); (ii) as a base space of a statistical mechanics (Gefen *et al* 1984, Suzuki 1983); (iii) as a structure on which we can define a real-space renormalisation procedure (Hilfer and Blumen 1984); and (iv) as a mathematical tool describing random figures with statistical self-similarity, e.g. the percolating clusters at threshold (Alexander and Orbach 1982, Rammal and Toulouse 1983).

A fractal is described by various metric properties, e.g. the Hausdorff dimension $\bar{d}$, which measures the density of (sites of) the fractal *embedded* in a Euclidean space. In the physical context, one is led to another kind of dimensionality and to the notion of the *intrisnic* property of a fractal, i.e. a property independent of the embedding (Rammal *et al* 1984a, b, Havlin and Nossal 1984, Vannimenus *et al* 1984, Havlin 1984). The spectral dimension $\tilde{d}$ is the first and the most important intrinsic characteristic of a fractal.

In this paper we will pay some attention to some basic mathematical properties of the spectral dimension. Our problems are:

(i) As mentioned above, $\tilde{d}$ is independent of the embedding. But we further expect that $\tilde{d}$ is determined by the global topology of a fractal. We have to prove rigorously that $\tilde{d}$ does not depend on the local structure of a fractal (§ 3).

(ii) In the constructive field theory based on the lattice regularisation, it is essential to take the continuum limit, i.e. a limit as the lattice spacing tends to zero (Fröhlich 1982). Consider the continuum limit of a diffusion process on a fractal and show the complete localisation in the continuum limit on such a fractal that $\tilde{d} < \bar{d}$ (§ 4).

(iii) Except for the regular lattices, any fractal for which $\tilde{d}$ has been evaluated satisfies $\tilde{d} < \bar{d}$. If $\tilde{d} = \bar{d}$, the fractal will be infinitely ramified. Apply the bond-moving approximation (Migdal–Kadanov renormalisation) (Burkhardt 1982) to a Sierpinski carpet and obtain an approximate value of $\tilde{d}$ (§ 5).

Our arguments are based on the wire version of the diffusion problem (Alexander 1983). The solution is written as a two-point function of a free field (§ 2). Our consideration of the wire problem is essential for the study of the continuum limit. Furthermore, it enables us to utilise the dependence of the density on the lattice spacing. In the appendix, basic estimates for free fields are shown. In this paper we only consider fractals for which $\tilde{d} < 2$ holds.

2. Solution of the wire problem

We consider a connected network Σ which has countably many vertices. Let V and B be the set of all vertices and the set of all bonds (wires) of Σ, respectively. We assume the following properties for Σ:

($\Sigma - 1$) The length a_{nm} of a bond $(n, m) \in B$ is uniformly bounded from below, i.e. we can find $\varepsilon > 0$ so that

$$a_{nm} > \varepsilon \qquad (n, m) \in B. \tag{2.1}$$

($\Sigma - 2$) Each vertex is connected to a uniformly bounded number of bonds, i.e.

$$\sum_{m(n)} 1 < N \qquad n \in V \tag{2.2}$$

for a constant N independent of $n \in V$, where the summation is taken over all $m \in V$ such that $(n, m) \in B$.

In this paper we call a network with the above properties simply a *network*.

Choose an origin $0 \in V$ arbitrarily. Then diffusion on Σ from 0 is described by

$$(\partial/\partial t) u(t, x) = \Delta u(t, x) \qquad t > 0 \text{ on bonds} \tag{2.3}$$

$$u(0, x) = 0 \qquad \text{on bonds} \tag{2.4}$$

$$\sum_{m(n)} j_{n \to m}(t) = \delta_{n0} \delta(t) \qquad n \in V, t \geq 0 \tag{2.5}$$

where $-j_{n \to m}(t)$ gives the gradient $(\partial/\partial x) u(t, x)$ at $x = n \in V$ along the direction from n to m. The essential point of the problem is to determine the density $u(t, n)$ at each vertex $n \in V$. We denote by $\sim$ the Laplace transform with respect to the time t and the dual variable is denoted by p. Put $G_n(p) = \tilde{u}(p, n)$ for $n \in V$. The following lemma is given by Alexander (1983) without proof.

Lemma 2.1. For $(n, m) \in B$, we have

$$\tilde{j}_{n \to m}(p) = [p^{1/2}/\tanh(a_{nm} p^{1/2})] G_n(p) - [p^{1/2}/\sinh(a_{nm} p^{1/2})] G_m(p). \tag{2.6}$$

Proof. We consider diffusion on an interval $[0, a]$ for $a > 0$:

$$(\partial/\partial t) u(t, x) = \Delta u(t, x) \qquad t > 0, x \in (0, a) \tag{2.7}$$

$$u(0, x) = 0 \qquad x \in (0, a) \tag{2.8}$$

$$u(t, 0) = \alpha(t) \qquad t > 0 \tag{2.9}$$

$$u(t, a) = \beta(t) \qquad t > 0 \tag{2.10}$$

for prescribed (non-negative) functions $\alpha(t)$ and $\beta(t)$ with a suitable condition. Put, for $t>0$ and $x, y \in [0, a]$,

$$K(t,x,y)=\tfrac{1}{2}(\pi t)^{-1/2}\exp[-(x-y)^2/4t]$$

$$U(t,x,y)=\sum_{m=-\infty}^{\infty}[K(t,x,y+2ma)-K(t,x,-y+2ma)].$$

Then the solution of (2.7)-(2.10) is written as

$$u(t,x)=\int_0^t(\partial/\partial y)U(t-s,x,0)\alpha(s)\,\mathrm{d}s-\int_0^t(\partial/\partial y)U(t-s,x,a)\beta(s)\,\mathrm{d}s.$$

By a straightforward calculation we obtain

$$\tilde{j}_{0\to a}(p)=[p^{1/2}/\tanh(ap^{1/2})]\tilde{\alpha}(p)-[p^{1/2}/\sinh(ap^{1/2})]\tilde{\beta}(p)$$

where $j_{0\to a}(t)=-(\partial/\partial x)u(t,0)$. This proves the lemma.

By virtue of (2.6), we can rewrite (2.5) as

$$\sum_{m\in V}M_{nm}(p)G_m(p)=\delta_{n0}\qquad n\in V \tag{2.11}$$

where

$$M_{nm}(p)=\sum_{l(n)}[p^{1/2}/\tanh(a_{nl}p^{1/2})]\delta_{nm}-p^{1/2}/\sinh(a_{nm}p^{1/2}) \tag{2.12}$$

if $(n,m)\in \mathrm{B}$, and otherwise

$$M_{nm}(p)=\sum_{l(n)}[p^{1/2}/\tanh(a_{nl}p^{1/2})]\delta_{nm}. \tag{2.13}$$

Since the diagonal part M_{diag} of $M(p)$ is invertible for $p>0$ and $(I+M_{\mathrm{diag}}^{-1}M_{\mathrm{off\text{-}diag}})^{-1}$ exists for $p>0$, $M(p)$ is invertible for $p>0$. Then we obtain from (2.11)

$$G_n(p)=M_{n0}^{-1}(p)\qquad p>0. \tag{2.14}$$

This solves (2.3)-(2.5) essentially.

Next we regard $G_n(p)$ as the two-point function of a free field $(\phi_n)_{n\in V}$ on V. Since we can write formally

$$\phi M(p)\phi=\sum_{(n,m)\in B}[p^{1/2}/\sinh(a_{nm}p^{1/2})](\phi_n-\phi_m)^2 + \sum_{n\in V}\sum_{m(n)}p^{1/2}\tanh(\tfrac{1}{2}a_{nm}p^{1/2})\phi_n^2 \tag{2.15}$$

the matrix $M(p)$, $p>0$, satisfies the hypothesis $(F-1)$-$(F-3)$ in the appendix. Then, as is shown in the appendix, $\langle F(\phi)\rangle^{M(p)}$ is well defined by (A1) for $L=M(p)$ as the infinite volume limit, where $F(\phi)$ is a polynomial depending on the finite number of variables ϕ_n. For a matrix L of finite size, it holds that $\langle\phi_n\phi_m\rangle^L=L_{nm}^{-1}$. As an infinite volume version, we have

Proposition 2.2. For $p>0$ it holds that

$$G_n(p)=\langle\phi_n\phi_0\rangle^{M(p)}\qquad n\in V.$$

3. Topological invariance of the spectral dimension

The spectral dimension $\tilde{d}$ characterises the infrared behaviour of dynamical processes on a fractal (Rammal and Toulouse 1983). In this paper we define $\tilde{d}$ for a network Σ by the following asymptotic form of the solution $u(t, 0)$ of (2.3)-(2.5):

$$u(t,0) \sim t^{-\tilde{d}/2} \qquad t \to \infty. \tag{3.1}$$

This can be translated into the Laplace transform as

$$G_0(p) \sim p^{\tilde{d}/2-1} \qquad p \to +0 \tag{3.2}$$

if $\tilde{d} < 2$. For the d-dimensional Sierpinski gasket, $\tilde{d} = 2\log_{10}(d+1)/\log_{10}(d+3)$ (Rammal and Toulouse 1983), and for the square lattices, $\tilde{d} = 2$ (see also Ben-Avraham and Havlin 1983, Hilfer and Blumen 1984). In this section we are concerned with topological invariance of the spectral dimension. We say that two networks $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are *topologically equivalent* if $V^{(1)} = V^{(2)}$ and $B^{(1)} = B^{(2)}$, where $V^{(i)}$ and $B^{(i)}$ are the set of all vertices and the set of all bonds in $\Sigma^{(i)}$, respectively. Then we can prove

Theorem 3.1. Let $\Sigma^{(i)}$, $i = 1, 2$, be topologically equivalent networks satisfying

$$C_1 < a^{(2)}_{nm}/a^{(1)}_{nm} < C_2 \qquad (n, m) \in B^{(i)}, i = 1, 2 \tag{3.3}$$

for some constants C_1 and $C_2 > 0$. Suppose that $\Sigma^{(1)}$ has the spectral dimension $\tilde{d} < 2$. Then $\Sigma^{(2)}$ has the same spectral dimension $\tilde{d}$.

We write $B = B^{(1)} = B^{(2)}$ and $V = V^{(1)} = V^{(2)}$. To prove the theorem, we consider matrices $M^{(i)}(p)$, $i = 1, 2$, defined similarly as $M(p)$ in § 2: explicitly

$$\phi M^{(i)}(p)\phi = \sum_{(n,m)\in B} [p^{1/2}/\sinh(a^{(i)}_{nm}p^{1/2})](\phi_n - \phi_m)^2$$
$$+ \sum_{n\in V}\left(\sum_{m(n)} p^{1/2}\tanh(\tfrac{1}{2}a^{(i)}_{nm}p^{1/2})\right)\phi_n^2 \qquad i = 1,2. \tag{3.4}$$

Lemma 3.2. If $a^{(1)}_{nm} \leq a^{(2)}_{nm}$ for any $(n, m) \in$ B, it holds that

$$\log_{10}\langle\phi_0^2\rangle^{M^{(1)}} - \log_{10}\langle\phi_0^2\rangle^{M^{(2)}} \leq C \sup_{(n,m)\in B} (a^{(2)}_{nm} - a^{(1)}_{nm}) \qquad p \in (0, 1) \tag{3.5}$$

where C is independent of $p \in (0, 1)$.

Proof. We apply lemma A2 (ii) from the appendix to

$$J^{(i)}_{nm} = p^{1/2}/\sinh(a^{(i)}_{nm}p^{1/2}) \qquad (n, m) \in B$$
$$= 0 \qquad (n, m) \notin B$$
$$g^{(i)}_n = \sum_{m(n)} p^{1/2}\tanh(\tfrac{1}{2}a^{(i)}_{nm}p^{1/2}) \qquad n \in V$$

where $i = 1, 2$. By virtue of (2.1) and (2.2), we have

$$g^{(i)}_n \geq p^{1/2}\tanh(\tfrac{1}{2}\varepsilon p^{1/2})$$
$$|g^{(1)}_n - g^{(2)}_n| \leq \tfrac{1}{2}Np \sup_{(n,m)\in B} (a^{(2)}_{nm} - a^{(1)}_{nm}).$$

Then (A10) implies (3.5).

Proof of the theorem. First we prove the theorem when there exists a constant K such that

$$a_{nm}^{(i)} < K \qquad (n, m) \in B, \qquad i = 1, 2. \tag{3.6}$$

Put, for $(n, m) \in B$,

$$a_{nm}^{(0)} = (\varepsilon/K) a_{nm}^{(1)}$$
$$a_{nm}^{(3)} = (K/\varepsilon) a_{nm}^{(1)}.$$

Then it holds that

$$a_{nm}^{(0)} \leqslant a_{nm}^{(2)} \leqslant a_{nm}^{(3)} \qquad (n, m) \in B. \tag{3.7}$$

Let $\langle \phi_0^2 \rangle^{M^{(i)}}$, $i = 0, 3$, be defined by (3.4) for $i = 0, 3$. Then lemma 3.2, together with (3.7), implies

$$\log\langle \phi_0^2 \rangle^{M^{(0)}} - \log\langle \phi_0^2 \rangle^{M^{(2)}} \leqslant CK \tag{3.8}$$

and

$$\log\langle \phi_0^2 \rangle^{M^{(2)}} - \log\langle \phi_0^2 \rangle^{M^{(3)}} \leqslant CK^2/\varepsilon. \tag{3.9}$$

Obviously $\langle \phi_0^2 \rangle^{M^{(i)}}$, $i = 0, 1, 3$, share the same asymptotic behaviour as (3.2), i.e.

$$\lim_{p \to +0} \langle \phi_0^2 \rangle^{M^{(i)}} \sim p^{\tilde{d}/2-1} \qquad i = 0, 3. \tag{3.10}$$

Thanks to (3.8) and (3.9), (3.10) also holds for $i = 2$.

In general cases, we add new vertices on long bonds: on each bond $(n, m) \in B$ in each network $\Sigma^{(i)}$, $i = 1, 2$, we add $k-1$ vertices and divide the bond into k equal parts, where $k - 1 < a_{nm}^{(1)} \leqslant k$. We denote the resulting networks by $\Sigma'^{(i)}$, $i = 1, 2$, respectively. Then the length $b_{nm}^{(i)}$ of a bond (n, m) in $\Sigma'^{(i)}$ satisfies

$$\begin{aligned} &\min(\varepsilon, \tfrac{1}{2}) < b_{nm}^{(1)} \leqslant 1 \\ &\min(\varepsilon, \tfrac{1}{2}C_1) < b_{nm}^{(2)} \leqslant C_2. \end{aligned} \tag{3.11}$$

Thus the general case reduces to the case satisfying (3.6).

Since the spectral dimension $\tilde{d}$ reflects only the long distance behaviour of a fractal, the definition of $\tilde{d}$ should be independent of the choice of the origin 0. In fact, we have the rigorous proof using theorem 3.1.

Theorem 3.3. If a network Σ has a spectral dimension $\tilde{d} < 2$, then $\tilde{d}$ is independent of the choice of the origin $0 \in V$.

We first prove the following lemma on the decimation of spin.

Lemma 3.4. Let 1 be a site of Σ different from the origin and put $V_1 = \{n \in V: (n, 1) \in B\}$. We consider the network $\Sigma^{(1)}$ obtained by eliminating site 1 and the bonds connected to 1 and then adding bonds, (n, m), for $n, m \in V_1$ (unless $(n, m) \in B$) with arbitrary lengths. Then the spectral dimension of $\Sigma^{(1)}$ coincides with that of Σ if the latter exists and is smaller than 2.

Proof. Let B_1 be the set of all bonds $(n, m) \in B$ such that $n, m \in V_1 \cup \{1\}$. Thanks to theorem 3.1, we can trim the bonds $(n, m) \in B_1$ so that $a_{nm} = a$ for a constant $a > 0$

without changing the value of $\tilde{d}$. We now approximate the matrix $M(p)$ defined by (2.15) as follows. Put, for $n, m \in V$,

$$\begin{aligned} J'_{nm} &= 1/a && \text{if } (n, m) \in B_1 \\ &= p^{1/2}/\sinh(a_{nm}p^{1/2}) && \text{if } (n, m) \in B \backslash B_1 \\ &= 0 && \text{otherwise} \end{aligned} \tag{3.12}$$

and put, for $n \in V$,

$$\begin{aligned} g'_n &= ap && \text{if } n \in V_1 \cup \{1\} \\ &= \sum_{m(n)} p^{1/2}\tanh(\tfrac{1}{2}a_{nm}p^{1/2}) && \text{otherwise.} \end{aligned}$$

By means of J'_{nm} and g'_n, we define the symmetric matrix M' by

$$\phi M'\phi = \tfrac{1}{2} \sum_{n,m \in V} J'_{nm}(\phi_n - \phi_m)^2 + \sum_{n \in V} g'_n \phi_n^2. \tag{3.13}$$

Then it holds that

$$\begin{aligned} &J'_{nm} - p^{1/2}/\sinh(a_{nm}p^{1/2}) = \mathrm{O}(p) \\ &g'_n - \sum_{m(n)} p^{1/2}\tanh(\tfrac{1}{2}a_{nm}p^{1/2}) = \mathrm{O}(p) \\ &g'_n > \min(ap, p^{1/2}\tanh(\tfrac{1}{2}\varepsilon p^{1/2})) = \mathrm{O}(p) \end{aligned}$$

uniformly in $n, m \in V$. Therefore we can use (A9) in the appendix to obtain

$$|\log\langle\phi_0^2\rangle^M - \log\langle\phi_0^2\rangle^{M'}| < C \tag{3.14}$$

where $C > 0$ is independent of $p \in (0, 1)$.

We next decimate the degree of freedom ϕ_1, i.e. carry out the integrations with respect to ϕ_1 in $\langle\phi_0^2\rangle^{M'}$. The resulting expression is again a free field correlation:

$$\langle\phi_0^2\rangle^{M'} = \langle\phi_0^2\rangle^{M''}.$$

Here M'' is defined by

$$\phi M''\phi = \tfrac{1}{2} \sum_{n,m \in V\backslash\{1\}} J''_{nm}(\phi_n - \phi_m)^2 + \sum_{n \in V\backslash\{1\}} g''_n \phi_n^2$$

where J''_n and g''_n are given by the following equality:

$$\begin{aligned} J''_{nm} &= \frac{1}{a}\left(1 + \frac{1}{a^2p + N_1}\right) && \text{if } n, m \in V_1 \text{ and } (n, m) \in B_1 \\ &= \frac{1}{a(a^2p + N_1)} && \text{if } n, m \in V_1 \text{ and } (n, m) \notin B_1 \\ &= J'_{nm} && \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} g''_n &= ap\left(1 + \frac{1}{a^2p + N_1}\right) && \text{if } n \in V_1 \\ &= g'_n && \text{otherwise} \end{aligned}$$

where $N_1 = \Sigma_{n(1)} 1$.

Next we consider the network $\Sigma^{(1)}$. Here we assign to each bond (n, m) in $\Sigma^{(1)}$ the length $a_{nm}^{(1)}$ given by

$$a_{nm}^{(1)} = \frac{N_1+1}{N_1} a \qquad \text{if } (n, m) \in B_1,$$

$$= \frac{a}{N_1} \qquad \text{if } n, m \in V_1 \text{ and } (n, m) \notin B_1$$

$$= a_{nm} \qquad \text{otherwise.}$$

Then lemma A2 (ii) implies that

$$|\log\langle\phi_0^2\rangle^{M^{(1)}} - \log\langle\phi_0^2\rangle^{M'}| < C' \qquad p \in (0, 1)$$

where $M^{(1)}$ is given by (2.15) with a_{nm} replaced by $a_{nm}^{(1)}$, and C' is a constant independent of p. This, together with (3.14), implies

$$|\log\langle\phi_0^2\rangle^{M^{(1)}} - \log\langle\phi_0^2\rangle^{M}| < C'' \qquad p \in (0, 1). \tag{3.15}$$

Thus we complete the decimation of the degree of freedom ϕ_1 and obtain a diffusion process on $\Sigma^{(1)}$. (3.15) shows that Σ and $\Sigma^{(1)}$ have the same value of $\tilde{d}$. Then thanks to theorem 3.1 we have the lemma.

Theorem 3.3 is a simple consequence of the following lemma, implying that we can move the origin to an adjacent site without varying the value of $\tilde{d}$.

Lemma 3.5. Let $(1, 2)$ be a bond of a network Σ with $\tilde{d} < 2$ and let Σ_0 be the network obtained from Σ by setting $a_{12} = 0$. Then Σ_0 has the same spectral dimension $\tilde{d}$.

Proof. First we consider the case $\Sigma_{m(1)} 1 = 1$. If site 1 is not the origin 0, lemma 3.4 yields lemma 3.5. In case site 1 is at the origin 0, thanks to the assumption $\Sigma_{m(1)} 1 = 1$, we can carry out the integrations with respect to $\phi_0 = \phi_1$ in $\langle\phi_0^2\rangle^M$ and obtain

$$\langle\phi_0^2\rangle^M = C_1(p)\langle\phi_2^2\rangle^{M_0} + C_2(p)$$

where $0 \ll C_j(p) \ll \infty$, $j = 1, 2$, for $0 < p < 1$, and M_0 is the matrix describing the diffusion on Σ_0. This implies the lemma.

Next we deal with the case

$$\sum_{m(i)} 1 > 1 \qquad i = 1, 2. \tag{3.16}$$

In this case, the constant C in (3.5) can be chosen independently of p and $a_{12}^{(j)}$, $j = 1, 2$. Therefore, if we write $\langle\cdot\rangle^M = \langle\cdot\rangle_a$ as a function of $a = a_{12}$, we have

$$\log\langle\phi_0^2\rangle_{a'} - \log\langle\phi_0^2\rangle_a < C_3 \tag{3.17}$$

where $0 < a' < a < 1$ and C_3 is independent of a', a and $p \in (0, 1)$. Let $a' \to 0$ in (3.17). Then we obtain

$$\log\langle\phi_0^2\rangle^{M_0} - \log\langle\phi_0^2\rangle^M \leqslant C_3 \tag{3.18}$$

since

$$\lim_{a' \to 0}\langle\phi_0^2\rangle_{a'} = \langle\phi_0^2\rangle^{M_0}. \tag{3.19}$$

(The equality (3.19) will be plausible because of the electric circuit analogue used in the proof of lemma A1 in the appendix. A rigorous proof of (3.19) is obtained by means of the equality

$$\lim_{\tau\to 0}(2(\pi\tau)^{1/2})^{-1}\exp(-(\phi_1-\phi_2)^2/4\tau)=\delta(\phi_1-\phi_2)$$

which works under the condition (3.16).)

Next, in the original network Σ, assuming that site 1 is not the origin, we decimate the site 1. Lemma 3.4 implies

$$|\log\langle\phi_0^2\rangle^M-\log\langle\phi_0^2\rangle^{M^{(1)}}|<C_4 \tag{3.20}$$

where $M^{(1)}$ is the matrix describing the diffusion on $\Sigma^{(1)}$, and C_4 is independent of $p\in(0,1)$. Note that the network $\Sigma^{(1)}$ is obtained from Σ_0 by varying the lengths of bonds connected to the site $1=2$ and, if necessary, by adding finite bonds. The former procedure leaves $\tilde{d}$ unchanged (theorem 3.1), while the latter does not decrease $\tilde{d}$ as is shown by the same method as lemma 3.2. Then we have

$$\log\langle\phi_0^2\rangle^{M^{(1)}}-\log\langle\phi_0^2\rangle^{M_0}<C_5. \tag{3.21}$$

Combining (3.18), (3.20) and (3.21), we obtain

$$|\log\langle\phi_0^2\rangle^M-\log\langle\phi_0^2\rangle^{M_0}|<C_6 \tag{3.22}$$

which proves the lemma.

Successive applications of (3.22) yield theorem 3.3.

We now proceed to show that $\tilde{d}$ of a network Σ is determined by its global topology. In order to formulate the notion of global topology, we prepare some new terminology. Let Σ_1 be a *sub-network* of Σ, i.e. a network obtained by eliminating some sites and bonds of Σ. A site n of Σ_1 is called an *interior point* of Σ_1 if any bond (n, m) of Σ is also a bond of Σ_1. Otherwise it is called a *boundary point.* We say that a sub-network Σ_1 is a *block* of Σ if Σ_1 has finite sites and if any pair of its boundary points can be connected with one another by a bond in Σ_1 directly or by a series of bonds in Σ_1 only through interior points of Σ_1. Let Σ_α, $\alpha=1,2,\ldots$, be blocks of Σ containing a uniformly bounded number of sites and having the property that, if for $\alpha\neq\beta$, Σ_α and Σ_β have a site n in common, n is a boundary point of Σ_α and Σ_β. In each block Σ_α, we eliminate all interior points in it and all bonds connected to them. The resulting network Σ^* is called a *skeleton* of Σ. We say that networks $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are *globally equivalent* if we can extract skeletons $\Sigma^{(i)*}$ of $\Sigma^{(i)}$ for $i=1,2$ in such a way that the extracted skeletons are topologically equivalent. Then we have the following theorem.

Theorem 3.6. Let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ be globally equivalent networks with bonds of uniformly bounded lengths. If $\Sigma^{(1)}$ has a spectral dimension $\tilde{d}<2$, then $\Sigma^{(2)}$ has the same spectral dimension $\tilde{d}$.

Proof. Let $\Sigma^{(i)*}$, $i=1,2$, be topologically equivalent skeletons of $\Sigma^{(i)}$, $i=1,2$, respectively, and let $\Sigma_\alpha^{(i)}$, $\alpha=1,2,\ldots$, $i=1,2$, be the blocks defining the skeletons $\Sigma^{(i)*}$, $i=1,2$, respectively. Thanks to theorem 3.3 we can assume that the origin of $\Sigma^{(i)}$, $i=1,2$, lies on the skeleton $\Sigma^{(i)*}$, $i=1,2$, respectively, and that the origins occupy the corresponding sites in each network. In the same way as in the proof of lemma 3.4, we decimate all interior points in all blocks $\Sigma_\alpha^{(i)}$, $\alpha=1,2,\ldots$, $i=1,2$. The resulting networks $\Sigma'(i)$, $i=1,2$, are topologically equivalent, and the length of their bonds are uniformly bounded from below and above. Then, applying theorem 3.1 to $\Sigma'^{(1)}$ and $\Sigma'^{(2)}$, we have the theorem.

4. Continuum limit

On the fractal for which $\tilde{d} < \bar{d}$ holds, diffusion exhibits an anomalous behaviour (Rammal and Toulouse 1983) (see (4.10)). Intuitively the anomaly causes the localisation of a random walker. In this section we shall explain the localisation by the analysis of the continuum limit of the free field on the fractal.

Consider a network Σ embedded in $\mathbb{R}^d$ in such a way that each bond is realised as a line segment in $\mathbb{R}^d$ and that the origin of Σ is located at $0 \in \mathbb{R}^d$. We assume the self-similarity of Σ, i.e. $\Sigma \subset \beta\Sigma$ for some $0<\beta<1$, where $\beta\Sigma$ denotes the reduction of Σ by a factor β with 0 as a centre. We are concerned with the diffusion process on $\beta^k\Sigma$ in the limit as $k\to\infty$, i.e. the continuum limit.

For $k=1,2,\ldots$, we denote by $M(k;p)$ the matrix defined by (2.15) with a_{nm} replaced by $\beta^k a_{nm}$. Let us consider the quantity

$$G_x^{(k)}(\delta;p) = \beta^{k(1-\delta)}\langle\phi_0\phi_{\beta^{-k}x}\rangle^{M(k;p)} \tag{4.1}$$

where $x\in\mathbb{R}^d$ is fixed so that we find a vertex of $\beta^k\Sigma$ at $x\in\mathbb{R}^d$ for a sufficiently large k: the vertex is denoted by $\beta^{-k}x$, which indicates the distance from 0 measured in units of the lattice. The factor $\beta^{k(1-\delta)}$ on the right-hand side of (4.1) renormalises the *wavefunction* ϕ_n. The canonical renormalisation corresponds to the value $\delta=\bar{d}$. This is a straightforward extension to a fractal of the renormalisation of the free field on the square lattice. We shall study the limit of $G_x^{(k)}(\delta;p)$ as $k\to\infty$.

We first consider the case $x=0$.

Proposition 4.1. If Σ has the spectral dimension $\tilde{d}<2$, then it holds that

$$G_0^{(k)}(\delta;p) \sim \beta^{k(\tilde{d}-\delta)} \qquad k\to\infty \tag{4.2}$$

for a fixed $p>0$.

Proof. Since each entry of $M(p)$ defined by (2.15) is a function of $a_{nm}p^{1/2}$ multiplied by $p^{1/2}$, $\langle\phi_0^2\rangle^{M(k;p)}$ is a function of $\beta^k p^{1/2}$ multiplied by $p^{-1/2}$:

$$\langle\phi_0^2\rangle^{M(k;p)} = p^{-1/2}h(\beta^k p^{1/2}). \tag{4.3}$$

Since $\beta^k\Sigma$ has the spectral dimension $\tilde{d}$ for each $k=1,2,\ldots$, we have

$$\langle\phi_0^2\rangle^{M(k;p)} \sim p^{\tilde{d}/2-1} \qquad p\to 0. \tag{4.4}$$

Comparing the right-hand sides of (4.3) and (4.4), we find the asymptotic form $h(\xi)\sim\xi^{\tilde{d}-1}$ as $\xi\to 0$. Then (4.3) implies

$$\langle\phi_0^2\rangle^{M(k;p)} \sim \beta^{k(\tilde{d}-1)}p^{\tilde{d}/2-1} \tag{4.5}$$

for small p and large k. This proves (4.2).

In particular, if we assume $\tilde{d}<\bar{d}$, $G_0^{(k)}(\bar{d};p)\to\infty$ as $k\to\infty$ for any $p>0$. This indicates the absence of diffusion in the continuum limit.

Next we discuss the case $x \neq 0$. We need some further assumptions. Suppose that all bonds in Σ have equal length a and that Σ satisfies the following property.

($\Sigma - 3$) Let $L(\mu)$, $\mu > 0$, be the matrix defined by

$$\phi L(\mu)\phi = \sum_{(n,m)\in B} (\phi_n - \phi_m)^2 + \mu^2 \sum_{n \in V} N_n \phi_n^2$$

where $N_n = \Sigma_{m(n)} 1$. We write $\langle \cdot \rangle_\mu = \langle \cdot \rangle^{L(\mu)}$. Then, for large $|n|$, it holds that

$$\langle \phi_0 \phi_n \rangle_\mu / \langle \phi_0^2 \rangle_\mu \sim (\mu^* |n|)^\alpha \exp(-\mu^* |n|) \tag{4.6}$$

for some constants α and μ^*, where $|n|$ denotes the distance between the sites n and 0 measured in units of the lattice. Here, α is independent of μ and satisfies

$$\alpha > -2 - \bar{d} \tag{4.7}$$

while μ^* depends on μ as

$$\mu^* \sim \mu^\nu \tag{4.8}$$

for small $\mu > 0$, where $\nu \in \mathbb{R}$ is independent of μ.

As is well known, the square lattice satisfies ($\Sigma - 3$) for $\alpha = -\frac{1}{2}$ and $\nu = 1$. Under these assumptions, we shall discuss the continuum limit for $x \neq 0$ by rough arguments.

First we derive the relation

$$\nu = \tilde{d}/\bar{d}. \tag{4.9}$$

Consider a random walk $x(t)$, $t \geqslant 0$, on Σ starting at 0. As is shown by Rammal and Toulouse (1983), the mean square radius $\langle x(t)^2 \rangle$ is asymptotically written as

$$\langle x(t)^2 \rangle \sim t^{\tilde{d}/\bar{d}} \qquad t \to \infty. \tag{4.10}$$

On the other hand, using the solution $u(t, x)$ of (2.3)-(2.5), we can write the mean square radius:

$$\langle x(t)^2 \rangle = \int_\Sigma |x|^2 u(t, x)\, \mathrm{d}x.$$

This implies

$$p^{-\tilde{d}/\bar{d}-1} \sim \int_\Sigma |x|^2 \tilde{u}(p, x)\, \mathrm{d}x \qquad p \to 0. \tag{4.11}$$

We approximate the right-hand side of (4.11) as follows: for each $x \in \Sigma$, we choose some vertex $n \in V$ near to x, and replace $\tilde{u}(p, x)$ by $\tilde{u}(p, n) = \langle \phi_0 \phi_n \rangle^{M(p)}$. Next we approximate the action $\phi M(p) \phi$ by

$$\phi M_0(p) \phi = \sum_{(n,m)\in B} (1/a)(\phi_n - \phi_m)^2 + \sum_{n \in V} \tfrac{1}{2} a p N_n \phi_n^2.$$

Since we are going to take the continuum limit, we shall use the above approximation only for small a. In such a case $M(p) - M_0(p)$ is small. Then we have

$$\begin{aligned} \tilde{u}(p, n) &\sim \langle \phi_0 \phi_n \rangle^{M_0} \\ &= a \langle \phi_0 \phi_n \rangle_{a(p/2)^{1/2}}. \end{aligned}$$

Therefore it holds that

$$\tilde{u}(p, n)/\tilde{u}(p, 0) \sim \langle \phi_0 \phi_n \rangle_{a(p/2)^{1/2}} / \langle \phi_0^2 \rangle_{a(p/2)^{1/2}}. \tag{4.12}$$

This, combined with (4.6) and (3.2), implies

$$\tilde{u}(p, n) \sim (\mu^*|n|)^\alpha \exp(-\mu^*|n|) p^{\bar{d}/2-1}$$

for large $|n|$ and small p, where

$$\mu^* \sim (a^2 p)^{\nu/2} \qquad p \to 0.$$

Lastly we approximate the volume element:

$$\int_\Sigma \mathrm{d}x \simeq \int_0^\infty \mathrm{d}r\, r^{\bar{d}-1}$$

where $r = |x|$. Using these approximations, we rewrite (4.11) as

$$p^{-\tilde{d}/\bar{d}-1} \sim a^{\nu\alpha} p^{(\nu\alpha+\bar{d})/2-1} \int_0^\infty r^{1+\alpha+\bar{d}} \exp(-\mu^* r)\, \mathrm{d}r.$$

This, together with (4.7) and (4.8), implies

$$p^{-\tilde{d}/\bar{d}-1} \sim a^{-\nu(\bar{d}+2)} p^{-\nu(\bar{d}+2)/2+\bar{d}/2-1} \qquad p \to 0$$

from which (4.9) follows.

By virtue of (4.9), we can easily derive the fact that diffusion does not occur in the continuum limit if $\tilde{d} < \bar{d}$ and $\tilde{d} < 2$, i.e.

$$\lim_{k\to\infty} G_x^{(k)}(\delta; p) = 0 \qquad p > 0, x \neq 0 \tag{4.13}$$

for any choice of $\delta \in \mathbb{R}$. Now fix $p > 0$ arbitrarily. We use (4.12) (for a replaced by β^k) and (4.6) (for x replaced by $\beta^{-k}x$) to obtain

$$\langle \phi_0 \phi_{\beta^{-k}x} \rangle^{M(k;p)} / \langle \phi_0^2 \rangle^{M(k;p)} \sim (\mu_k^* \beta^{-k}|x|)^\alpha \exp(-\mu_k^* \beta^{-k}|x|)$$

where $\mu_k^* \sim \beta^{k\nu} = \beta^{k\tilde{d}/\bar{d}}$ as $k \to \infty$. This, combined with (4.5), implies

$$G_x^{(k)}(\delta; p) \sim \beta^{k[\alpha(\tilde{d}/\bar{d}-1)+\tilde{d}-\delta]} |x|^\alpha \exp(-\beta^{k(\tilde{d}/\bar{d}-1)}|x|)$$

as $k \to \infty$. Letting $k \to \infty$, we obtain (4.13).

5. Bond-moving approximation for the Sierpinski carpet

As is roughly shown in the preceding section, if diffusion occurs in the continuum limit, the equality $\tilde{d} = \bar{d}$ holds. However we have no example satisfying $\tilde{d} = \bar{d}$ except for the regular lattices. If such an example exists, it will be an infinitely ramified fractal. For some class of finitely ramified fractals, the spectral dimension $\tilde{d}$ can be evaluated by exact renormalisation group schemes (Hilfer and Blumen 1984, see also Ben-Avraham and Havlin 1983). However, for infinitely ramified fractals, the value of $\tilde{d}$ is still unknown. Therefore it will not be meaningless to obtain an approximate value of $\tilde{d}$ for an infinitely ramified fractal.

We consider a Sierpinski carpet (SC) constructed in the following way: we start with a square of side length $a > 0$ (figure 1(a)). We assemble the eight copies of the square into a large square of side length $3a$ (figure 1(b)). The procedure is then repeated for the larger square and iterated infinitely.

Gefen *et al* (1984) studied the Ising model and resistor networks on the SC (and on its variations) by approximate renormalisation group schemes, i.e. the bond-moving

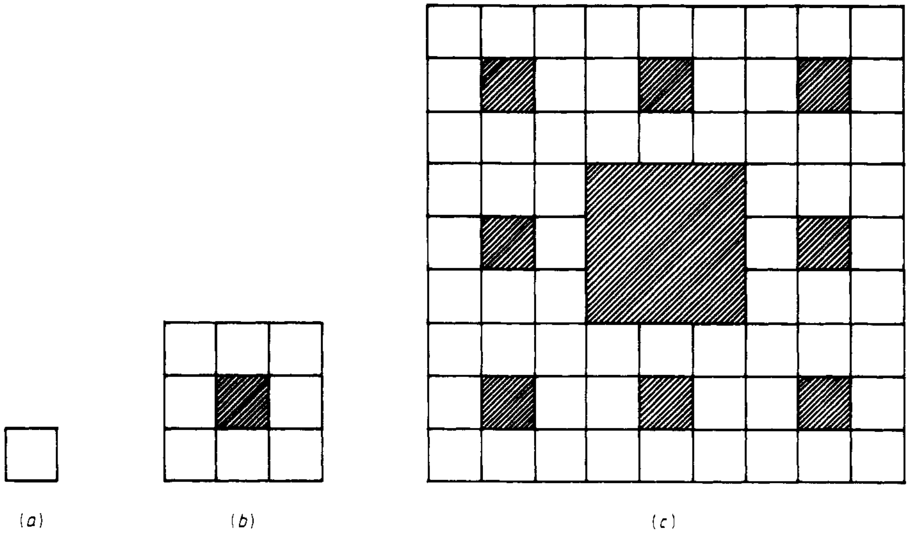

Figure 1. The Sierpinski carpet as a network: the initial square and the first two construction stages are shown.

approximation (Migdal-Kadanov renormalisation) (Burkhardt 1982). In particular, they obtained conductance exponents $\tilde{\zeta}$ approximately. Therefore, if we use the conjectured relation (Rammal and Toulouse 1983)

$$\tilde{\zeta} = (\bar{d}/\tilde{d})(2-\tilde{d}) \tag{5.1}$$

we can evaluate $\tilde{d}$. However, the recursion relation (4.1) in Gefen *et al* (1984) is inconsistent with the bond-moving procedure used in § 3 of the same paper. In fact, the latter gives equation (5) in Gefen *et al* (1983) instead of (4.1) in Gefen *et al* (1984). Using this equation (5), we obtain $\tilde{\zeta} \simeq \log_{10} \frac{7}{6}/\log_{10} 3 \simeq 0.1403$ for our SC, which together with (5.1) implies

$$\tilde{d} \simeq \frac{2 \log 8}{\log (28/3)} \simeq 1.862. \tag{5.2}$$

In this section we shall show that the same value of $\tilde{d}$ as (5.2) is obtained directly by the bond-moving approximation without using (5.1).

Consider the free field on SC with the action

$$\phi L \phi = \sum_{(n,m)\in B} J_{nm}(\phi_n - \phi_m)^2 + \sum_{n\in V} \sum_{m(n)} g_{nm}\phi_n^2$$

where we put $J_{nm} = K$ and $g_{nm} = h$ if (n, m) is a bond which borders the exterior of the SC or borders a courtyard (the hatched square in figure 1) made out at one of the stages of the construction. Otherwise, we put $J_{nm} = J$ and $g_{nm} = g$ for $J, K, g, h > 0$. In the following we write $\langle \cdot \rangle^L = \langle \cdot \rangle_{JKgh}$ and denote the corner of SC by 0.

We apply the bond-moving technique to SC as in § 3 of Gefen *et al* (1984): we decimate such degrees of freedom ϕ_n that n is a corner of one of the smallest courtyards (hatched squares of side length a) or that n is one of the nearest-neighbour sites of the corners. After the decimation, the resulting network is the SC of a lattice spacing $3a$, and we obtain

$$\langle \phi_0^2 \rangle_{JKgh} \simeq \langle \phi_0^2 \rangle_{J'K'g'h'} \tag{5.3}$$

where

$$J' = \frac{9J^2(J+2K)}{(3J+4g+2h)(5J+4K+4g+2h)} \tag{5.4a}$$

$$K' = \frac{2K(J+K)^2}{(J+K+g+3h)(J+5K+g+3h)} \tag{5.4b}$$

$$g' = 3g + \frac{3J(4g+2h)}{3J+4g+2h} \tag{5.4c}$$

$$h' = g + h + \frac{(J+K)(g+3h)}{J+K+g+3h}. \tag{5.4d}$$

Of course we can iterate the procedure any number of times. However the approximation (5.3) will be good only for $J,K \gg g,h$. The reason is understood intuitively by the electric circuit analogue used in the proof of lemma A1. If J and K are small, a resistor on a bond insulates nearest-neighbour sites. Then the procedure to move resistors across the insulators will cause a fatal change of character.

Let us apply the above approximation for a free field to the problem of diffusion on the SC. Now fix $p>0$. We start with the SC of lattice spacing $a=3^{-N}$ for a sufficiently large N for which $3^{-N}p^{1/2} \ll 1$. In this case the action $\phi M(p)\phi$ defined by (2.15) is well approximated by

$$\phi L \phi = \sum_{(n,m)\in B} 3^N(\phi_n - \phi_m)^2 + \sum_{n\in V} \tfrac{1}{2}3^{-N}p\phi_n^2$$

where we used lemma A2 (i). Then we put

$$J_0 = K_0 = 3^N$$

$$g_0 = h_0 = \tfrac{1}{2}3^{-N}p$$

and define J_k, K_k, g_k, h_k, $k=1,2,\ldots,$ inductively by (5.4).

Since we use (5.3) only for J, $K \gg g,h$, we approximate (5.4) for $J,K \gg g,h$:

$$J' = \frac{J(3J+6K)}{5J+4K} \tag{5.5a}$$

$$K' = \frac{2K(J+K)}{J+5K} \tag{5.5b}$$

$$g' = 7g + 2h \tag{5.5c}$$

$$h' = 2g + 4h. \tag{5.5d}$$

A primitive analysis shows that the ratio K_n/J_n rapidly converges to $\frac{1}{2}$ and that

$$J_n, K_n \simeq (6/7)^n 3^N \qquad 1 \ll n \ll \infty.$$

On the other hand, (5.5c) and (5.5d) imply

$$g_n, h_n \simeq 8^n 3^{-N} p \qquad 1 \ll n \ll \infty.$$

Note that J_n and K_n decrease whereas g_n and h_n increase. We now choose $\lambda \gg 1$ arbitrarily and denote by n^* the value of n such that $J_n/g_n \simeq \lambda$, i.e. $n^* =$

$N \log 9/\log(28/3) \equiv \alpha N$. Put

$$g^* = g_{n^*} = 8^{\alpha N} 3^{-N} p. \tag{5.6}$$

Then successive applications of (5.3) imply

$$\begin{aligned} \langle \phi_0^2 \rangle_{J_0 K_0 G_0 H_0} &\simeq \langle \phi_0^2 \rangle_{\lambda g^*, \lambda g^*, g^*, g^*} \\ &= g^{*-1} \langle \phi_0^2 \rangle_{\lambda\lambda 11}. \end{aligned} \tag{5.7}$$

From (5.7) we can find the value of $\tilde{d}$: the asymptotic form (4.5) implies

$$\langle \phi_0^2 \rangle_{J_0 K_0 g_0 h_0} \sim 3^{-N(\tilde{d}-1)} \qquad N \to \infty. \tag{5.8}$$

Since $\langle \phi_0^2 \rangle_{\lambda\lambda 11}$ is independent of N, (5.7) and (5.8) show $g^* \sim 3^{N(\tilde{d}-1)}$. This, combined with (5.6), implies (5.2).

We conclude this section with the following remarks.

(i) We analysed above the massive (but almost massless) free fields, while the argument by Gefen *et al* (1984) deriving the values of $\tilde{\zeta}$ corresponds to the massless case. The agreement in the resulting values of $\tilde{d}$ being reasonable, its significance will rather be found in the confirmation of the conjecture (5.1) in the case of our SC.

(ii) As for the reliability of the Migdal-Kadanov renormalisation, we know little. We here try to test the procedure by applying it to the two-dimensional Sierpinski gasket obtained by connecting three midpoints on a regular triangle successively (Rammal and Toulouse 1983). It is well known that $\tilde{d} = 2\log 3/\log 5$ (Rammal and Toulouse 1983). Applying the Migdal-Kadanov renormalisation, we obtain $\tilde{d} \simeq 2\log 3/\log(9/2)$, which is larger than the exact value. Then we could expect that, for our SC, $\tilde{d} < 2\log 8/\log(28/3)$.

(iii) (5.2) gives $\tilde{d}/\bar{d} \simeq \log 9/\log(28/3) \simeq 0.9837$ since $\bar{d} = \log 8/\log 3 = 1.8928$. Then we expect that diffusion does not occur on the SC in the continuum limit as a prediction by the Migdal-Kadanov renormalisation. (The discussion in (ii) strengthens this expectation.)

6. Conclusion

In this paper we have analysed diffusion on a wire network. Assuming $\tilde{d} < 2$, we give a rigorous proof for the fact that the local geometry of a fractal is irrelevant to the spectral dimension $\tilde{d}$. For a fractal satisfying $\tilde{d} < \min(2, \bar{d})$, we roughly showed a complete localisation in the continuum limit under an assumption for the free field on the fractal. The bond-moving approximation predicts that diffusion does not occur on a Sierpinski carpet in the continuum limit. Whether there exists a fractal (other than the regular lattices, etc) satisfying $\tilde{d} = \bar{d}$ is unknown.

Acknowledgment

The author gratefully acknowledges stimulating discussions with T Hattori and A Yoshioka. He also expresses his thanks to K Hattori who pointed out a gap contained in the previous version of this work.

Appendix. Basic estimates for free fields

Let V be a lattice with countably many vertices. We consider a free field on V where the thermal expectation of a polynomial $F(\phi)$ is given by

$$\langle F(\phi)\rangle^L = \int F(\phi)\exp(-\tfrac{1}{2}\phi L\phi)\,\mathrm{d}\phi\left(\int \exp(-\tfrac{1}{2}\phi L\phi)\,\mathrm{d}\phi\right)^{-1} \tag{A1}$$

$$\phi L\phi = \tfrac{1}{2}\sum_{n,m\in V} J_{nm}(\phi_n-\phi_m)^2 + \sum_{n\in V} g_n\phi_n^2. \tag{A2}$$

First we prove the well definedness of $\langle F(\phi)\rangle^L$ as the infinite volume limit (for details, see Griffiths 1972). We assume

$(F-1)$ For $(n, m)\in V$, $J_{nm}\geq 0$.

$(F-2)$ For a fixed $n\in V$, $J_{nm}>0$ only for at most N vertices m, where N is independent of n.

$(F-3)$ There exists a constant $g>0$ such that $g_n>g$, $n\in V$.

The well definedness of $\langle F(\phi)\rangle^L$ for a monomial $F(\phi)$ follows from the superstability

$$\langle F(\phi)\rangle^L_{\text{finite volume}} < \text{constant independent of the volume} \tag{A3}$$

and the monotonicity of the left-hand side of (A3) with respect to the volume. The superstability (A3) is easily shown by the Gaussian (in)equality

$$\langle G(\phi)\phi_n\rangle^L_{\text{finite volume}} = \sum_{m\in V}\langle(\partial/\partial\phi_m)G(\phi)\rangle^L_{\text{finite volume}}\langle\phi_m\phi_n\rangle^L_{\text{finite volume}}. \tag{A4}$$

In fact, by means of (A4) we can write $\langle F(\phi)\rangle^L_{\text{finite volume}}$ as a polynomial of two-point correlations $\langle\phi_i\phi_j\rangle^L_{\text{finite volume}}$, each one of which converges to L^{-1}_{ij} as the volume tends to ∞. On the other hand, the monotonicity can be shown by the Griffiths inequality II (Griffiths 1972).

Next we establish

Lemma A1. Fix an origin $0\in V$ arbitrarily. Then we have the following estimates:

$$0\leq\langle\phi_0\phi_n\rangle^L\leq\langle\phi_0^2\rangle^L \qquad n\in V \tag{A5}$$

$$\sum_{n\in V}\langle\phi_0\phi_n\rangle^L\leq 1/g \tag{A6}$$

where L and $\langle\cdot\rangle^L$ are defined by (A1) and (A2), respectively, and we assume $(F-1)$-$(F-3)$.

Proof. We give here an intuitive proof by means of the electric circuit analogue (Straley 1980). We connect each vertex $n\in V$ with the ground by a resistor with conductivity g_n (Ω^{-1}). Furthermore, for every pair (n, m) of vertices of V such that $J_{nm}>0$, we connect them by a resistor with conductivity $J_{nm}(\Omega^{-1})$. If we apply a direct current of 1A (from the power supply) to the origin 0, we shall detect the voltage $\langle\phi_0\phi_n\rangle^L$ (V) at each vertex $n\in V$. Then (A5) is intuitively valid. Since the total current is $\Sigma_{n\in V}g_n\langle\phi_0\phi_n\rangle^L = 1$ (A), we obtain (A6).

Lemma A2. Let $L^{(a)}$, $a=1,2$, be the matrix defined by (A2) with J_{nm} and g_n replaced by $J_{nm}^{(a)}$ and $g_n^{(a)}$, $a=1,2$, respectively, which are assumed to satisfy $(F-1)$-$(F-3)$.

(i) Suppose that

$$\sum_{n(m)} |J_{nm}^{(1)} - J_{nm}^{(2)}| \leq K \qquad n \in V \tag{A7}$$

$$|g_n^{(1)} - g_n^{(2)}| \leq K' \qquad n \in V \tag{A8}$$

for some constants K and K'. Then it holds that

$$|\log \langle \phi_0^2 \rangle^{L^{(2)}} - \log \langle \phi_0^2 \rangle^{L^{(1)}}| \leq 2N(K+K')/g. \tag{A9}$$

(ii) If we assume $j_{nm}^{(2)} \leq J_{nm}^{(1)}$, n, $m \in V$, together with (A8), we have

$$\log \langle \phi_0^2 \rangle^{L^{(1)}} - \log \langle \phi_0^2 \rangle^{L^{(2)}} \leq K'/g. \tag{A10}$$

Proof. Put, for $t \in [0, 1]$,

$$J_{nm}(t) = (1-t)J_{nm}^{(1)} + tJ_{nm}^{(2)} \qquad n, m \in V$$

$$g_n(t) = (1-t)g_n^{(1)} + tg_n^{(2)} \qquad n \in V.$$

Then $J_{nm}(t) \geq 0$, $g_n(t) > g$ and, for each n, $J_{nm}(t)$ is non-zero only for at most $2N$ vertices m. Let $L(t)$ denote the matrix defined by (A2) with J_{nm} and g_n replaced by $J_{nm}(t)$ and $g_n(t)$, respectively. Interchanging the order of $\mathrm{d}/\mathrm{d}t$ and $\langle\ \rangle$, we have

$$(\mathrm{d}/\mathrm{d}t)\langle \phi_0^2 \rangle^{L(t)} = -\tfrac{1}{2}\langle \phi_0^2 \phi L'(t) \phi \rangle^{L(t)} + \tfrac{1}{2}\langle \phi_0^2 \rangle^{L(t)} \langle \phi L'(t) \phi \rangle^{L(t)}. \tag{A11}$$

We now note that the infinite volume version of (A4) implies

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle^{L(t)} = \langle \phi_a \phi_b \rangle^{L(t)} \langle \phi_c \phi_d \rangle^{L(t)} + \langle \phi_a \phi_c \rangle^{L(t)} \langle \phi_b \phi_d \rangle^{L(t)} + \langle \phi_a \phi_d \rangle^{L(t)} \langle \phi_b \phi_c \rangle^{L(t)}.$$

Using the above formula, we obtain from (A11)

$$(\mathrm{d}/\mathrm{d}t)\langle \phi_0^2 \rangle^{L(t)} = -\sum_{n,m \in V} L'_{nm}(t) \langle \phi_0 \phi_n \rangle^{L(t)} \langle \phi_0 \phi_m \rangle^{L(t)}. \tag{A12}$$

By means of (A5) and (A6) (for L replaced by $L(t)$), we can estimate the right-hand side of (A12):

$$|(\mathrm{d}/\mathrm{d}t)\langle \phi_0^2 \rangle^{L(t)}| \leq N(K+K')\langle \phi_0^2 \rangle^{L(t)} \sum_{n \in V} \langle \phi_0 \phi_n \rangle^{L(t)}$$

$$\leq (K+K')(2N/g)\langle \phi_0^2 \rangle^{L(t)}$$

where we used the estimate $|L'_{nm}(t)| \leq K + K'$. This proves (A9). If we assume $J_{nm}^{(1)} - J_{nm}^{(2)} \geq 0$, (A12) implies

$$(\mathrm{d}/\mathrm{d}t)\langle \phi_0^2 \rangle^{L(t)} \geq -\sum_{n \in V} g'_n(t)[\langle \phi_0 \phi_n \rangle^{L(t)}]^2$$

$$\geq -(K'/g)\langle \phi_0^2 \rangle^{L(t)}$$

which shows (A10).

References

Alexander S 1983 *Phys. Rev.* B **27** 1541–57
Alexander S and Orbach R 1982 *J. Physique Lett.* **43** 625–31
Ben-Avraham D and Havlin S 1983 *J. Phys. A: Math. Gen.* **16** L559–63

Burkhardt T W 1982 *Real-Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer) pp 33-56
Fröhlich J 1982 *Nucl. Phys.* B **200** [FS4] 281-96
Gefen Y, Aharony A and Mandelbrot B B 1984 *J. Phys. A: Math. Gen.* **17** 1277-89
Gefen Y, Meir Y, Mandelbrot B B and Aharony A 1983 *Phys. Rev. Lett.* **50** 145-8
Griffiths R B 1972 *Phase Transitions and Critical Phenomena* vol 1, ed C Domb and M S Green (New York: Academic) pp 7-109
Havlin S 1984 *Kinetics of Aggregation and Geration* ed F Family and H Gould (Amsterdam: North-Holland)
Havlin S and Nossal R 1984 *J. Phys. A: Math. Gen.* **17** L427-32
Hilfer R and Blumen A 1984 *J. Phys. A: Math. Gen.* **17** L783-9
Rammal R 1984 *Phys. Rep.* **103** 151-9
Rammal R, Angles d'Auriac J C and Benoit A 1984a *J. Phys. A: Math. Gen.* **17** L491-4
Rammal R and Toulouse G 1983 *J. Physique Lett.* **44** 13-22
Rammal R, Toulouse G and Vannimenus J 1984b *J. Physique* **45** 389-94
Straley J P 1980 *J. Phys. C: Solid State Phys.* **13** 2991-3002
Suzuki M 1983 *Prog. Theor. Phys.* **69** 65-76
Vannimenus J, Nadal J P and Martin H 1984 *J. Phys. A: Math. Gen.* **17** L351-6